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ON THE METHODS OF APPLIED MATHEMATICS

THE views expressed in this paper are not consciously derived from those of previous methodologists, but from an examination of the concrete processes of mathematical physics. As they are to that extent original and as, moreover, the form is different from that of the majority of logicians, no useful purpose would be served by reference to works in which it might be possible to discover the germs of some of these ideas.[1]

Nor is this essay in any way connected with the conflicting theories of knowledge or with controversies on the logical nature and philosophy of pure mathematics. Its special feature is the separation of the methodology of applied mathematics from these more speculative considerations, and the result is that the former stands out in a clearer light and attains greater relevance to the actual problems of mathematico-physical investigation.

Whatever may be our opinions concerning the ultimate nature or origin of mathematical reasoning, it would be impossible to deny that it is actually, practically, a series of abstractions. Though the axioms may be necessary, and the reasoning may possess intuitive and absolute validity, this validity is conditioned by the abstractions themselves, and the applicability of the conclusions to any particular existences in the object world is not necessarily implied in the process of mathematical reasoning.

Even in simple cases this can readily be illustrated. The simple arithmetical statement that $2 + 2 = 4$ is a definite unassailable truth of the conceptual order. The concrete application $2d + 2d = 4d$ would be false unless the objects to which it was applied were definite, separable, discrete units which admitted of addition. If, for example, we attempted to apply such arithmetical reasoning to the number of seas or gulfs in the Mediterranean, it would obviously give us a false, meaningless, or invalid result because there is no criterion by which we could say when and where one gulf was contained in or graded into another. The necessary discrete units are nowhere to be found.

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[1] The nearest approach to an anticipation will perhaps be found in Jevons's "Principles of Science," Chapter XXI. Between that statement and the following there are, however, a number of important differences.

It will thus be seen that it is possible, by too close an attention to the conceptual series, to make serious errors in the application of even the most elementary mathematical truths. According to such a strict logical rendering, simple addition is a "pure" mathematical process of absolute conceptual validity, compound addition is a mixed mathematical process dependent on the fact that we actually deal with perceptual discrete units to which the reasoning can be applied. In simple cases of this kind, it would be pedantic to insist on the distinction, but, for logical consistency, it is easy to point out that even here it is valid.

The example of mathematical reasoning in which the conceptual and perceptual elements are most entangled is geometry. This science, dealing as it does with existence under its simplest aspect of extension in space, gives us conceptual truths that are almost universally applicable to the perceptual world and can be verified to a high degree of approximation by experiments with lines drawn on paper. When applied to actual existences, such truths require some small qualifications which are necessitated by the structure of the substances used. The universal applicability of the truths of geometry depends on the perceptual fact that such qualifications are, as a rule, so small as to be practically negligible. Even in geometry, the modern development of fourth dimension and non-Euclidean applications shows that the conceptual series may outrun the perceptual and give us conclusions that have no valid meaning in the object world.

According to this view, the distinction between pure and applied mathematics is one of degree and of convenience. All mathematical reasoning is "pure," and all application is "mixed." In some cases, however, the applicability of the method is so obvious that it can be taken for granted; in others the discussion of the range within which the ideal construction is applicable to concrete instances is a difficult and complicated problem, only to be settled by prolonged and careful experiment, observation, and reasoning.

The starting-point of the present investigations can therefore be summarized in the following three propositions: (*a*) Mathematics is a series of abstractions, a conceptual series of truths, the validity and *a priori* certainty of which is self-contained and conditional on the ideal nature of the construction; (*b*) common knowledge and its outgrowth, experimental science, is a series of truths of a different order. Some may be *a priori*, but most are discovered by the intelligent observation of the object world; (*c*) the inter-relation of (*a*) and (*b*) is variable, and is dependent on the nature of the abstractions and of the facts of observation and experiment to which they can be applied.

We will now consider our third proposition more closely and see if we can discover any general rules and conditions which will throw light on the nature and necessary limits of this inter-relation. A number of these are immediately apparent. If, for purposes of calculation, we abstract from the totality of our experience certain general properties and work these into our conceptual series, it is obvious that the range within which our concepts admit of valid application depends on the range and certainty of these abstractions.

All Euclidean geometry is built up of a very few of these assumptions, which can be reduced to the fundamental axiom of quantity, Playfair's axiom, and some other self-evident truths. These, according to my own view, are *a priori* in the experience of the race; but, for those who still retain the preevolutionary empiricism of Mill, we can merely say that they are invariably consistent with experience. No one could doubt that for practical purposes their certainty is absolute. So far as the conceptual series includes these and no others, there is no practical limit to the length of the chain of reasoning that can be based on them. At any link in that chain, the conclusion could be verified by actual measurements which only vary from the ideal by the necessary limits of experimental error.

A few, a very few, physical axioms have this same degree of validity. In this class I should place the law of inertia, the conservation of matter (or inertia)[2] which is closely related to the former, and the conservation of energy. These are generally admitted to be of universal applicability, and, so far as mathematical reasoning is based on these, its validity would be as great as that of Euclidean geometry.

One or two other principles, while not *a priori*, possess an empirical validity practically universal. The law of gravitation is perhaps the best example of these, and here again there is no recognizable practical limit to the possibility of application to actual existences. But even here an element of uncertainty will creep in. A calculation based on the continual existence of any portion of ponderable matter would only have approximate validity if carried into the infinities of space and time. It is quite conceivable that matter may evolve from and dissolve into the ether of space. To admit this possibility we must postulate inertia as a property of the ether, but the possibility is there none the less. It is not likely that any actual calculation will be invalidated by such an error, but it well illustrates the empirical nature of mathematical investigations to point out that, at this early stage, we find a limit to its practical certainty.

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[2] Stated in this form, the principle admits the possibility of the evolution and dissolution of ponderable matter.

It will be seen that no series of physical calculations can possess the same range of applicability as ordinary Euclidean geometry. There is no geometry of mass. In theoretical mechanics we have no doubt a conceptual science which, so far as it is based on physical and geometrical axioms, would possess an equivalent theoretical certainty; yet the transference of the conclusions from the conceptual to the perceptual series contains numerous possibilities of error. Outside the limits of statics and dynamics, the validity of physical laws falls off rapidly. All the ordinary generalizations of physical science, laws of radiation with temperature, of conduction of heat, of relation between chemical and physical properties, of electric potential and current, all these, when they are not definitions in disguise, are avowedly empirical approximations applicable to actual existences only within reasonable and recognized limits. Any undue extension of these requires careful and critical treatment, and, if far removed from the possibility of verification, should not be regarded as definite and established truth.

So far we have dealt only with the certainty of the premises, the degree of validity *a priori* or otherwise of the abstractions from which the argument took its origin, and the consequent danger of error from the fact that these premises themselves correspond only imperfectly to actual existences in the object world. Any error on this score would imply a limit to the possibility of concrete applications. But the vast majority of mathematically established deductions not only imply the theoretical possibility of concrete application, but the statement that the abstractions provide an adequate correspondence to concrete reality.

We can readily illustrate this by reference to astronomy. All calculations of gravitational astronomy are not only dependent on the law of inverse squares and the continued existence of an invariable quantity of gravitating matter, but on what we may term a number of existential propositions concerning the actual system to which the mathematics is applied. These calculations also require the disputable and even false premises that the sun and planets are unchangeable in mass, suffering neither loss nor gain; that there are no sources of gravitational force other than the sun and the planets; and that there are no forces other than gravitation which can affect the relative position of celestial bodies. The last in particular is certainly wrong. Gravitational theory will not account for the position of the tail of a comet nor (probably) for the distribution of a spiral nebula, yet these astronomical calculations are the most accurate known to science.

The possibility of an extended series of astronomical calculations depends on the empirical fact of observation that all these sources of error are, for ordinary limits of space and time, so small as to be practically negligible. But, even here, any attempt to calculate the exact relative positions of the planets, say a million years ago or a million years hence, would rightly be received with some considerable degree of scepticism.[3]

All applications of statics and dynamics to actually existing bodies and forces are subject to a similar series of existential assumptions.

It will thus be seen that the possibilities of the interrelation of our conceptual and perceptual series is subject to two necessary limits: (1) That the abstractions on which the reasoning is based have, in a very small percentage of cases, absolute intuitive certainty or intrinsic accuracy; (2) that the concepts are never complete abstractions from reality, and the degree of completeness can only be determined empirically by observation and experiment. Of these two necessary limits, the second is, in normal cases, probably of the greatest practical importance; but both must be noted as theoretical sources of error.

An attempt to calculate the volume of a gas at 1,000 atmosphere by Boyle's law, or to discover the conditions of the atmosphere 500 miles above the earth's surface by the law of diffusion,[4] would undoubtedly come under the first heading. These principles are both empirical laws of limited applicability. A calculation of the age of the earth by secular cooling, or of the position of Halley's comet 5,000 years hence from its present velocity and position and from the gravitation of the sun and other calculable centers, would be invalid from the second cause. The abstractions in each of these cases would be entirely inadequate for the complexity of the problem to be solved.

For the remainder of this article, however, we shall not trouble further with these distinctions. It is possible that, if approached from a standpoint less practical than that of the present paper, they might be shown to be part of a larger principle. Whether this be so or not, we shall find that their practical importance will be increased or decreased by similar causes and conditions.

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[3] I lay no great stress on this suggestion which would require great mathematical analysis to follow to its conclusion; but is it not possible that Leverrier's calculations regarding the periodicity of the variations in the eccentricity of the earth's orbit may be invalid on this account? While his calculations are probably approximate in relation to the last glacial epoch, is it not possible that, through the multiplication of such minute errors as change of orbit due to solar tidal reaction, or gain of mass due to access of meteorites, the greater changes in the eccentricities of planetary orbits may not be periodic at all, but irregular and incalculable. If this suggestion be correct, it may throw light on the geologic problem of the irregular occurrence of glacial epochs, and Croll's hypothesis may be correct in spite of this difficulty.

[4] The law of diffusion depends on the straightness of the path of the individual molecules, which would be curved by gravity if molecular impact were infrequent.

Here is the next step in our investigation. Having discovered two necessary limits to the interrelation of the conceptual series of mathematical abstractions and the perceptual series of facts of observation and experiment, we must now endeavor to ascertain when and to what extent these limits are of definite practical importance. This it is impossible to accomplish completely in general terms. The necessary limits of mathematical investigation vary in incalculable ways with the nature of the special investigation. But, though such an aim can only be accomplished to a limited extent, it is, I believe, possible to lay down general rules, some of which we shall find are immediately applicable to the subject of this essay.

Such general rules will be best elucidated by the critical analysis of special concrete instances in the light of the foregoing principles.

Let us take, as an example, the simple kinematical formula for calculating the distance covered by a falling body ($s = \frac{1}{2}gt^2$). A very elementary knowledge of the facts will show us that this useful little formula will only serve our purpose within a very limited range. Immediately one of the variables is increased beyond a definite limit, a correction must be introduced for the resistance of the air. This, indeed, is so difficult that I doubt whether it has been attempted. But a moderate approximation would imply a complicated differential, including (at least) a variable for the density of the atmosphere, another for the velocity of the body, and (assuming a definite shape) a constant for the ratio between the mass of the body and its surface area.

Let us imagine these difficulties to have been overcome, or the calculation to refer to a planet without an atmosphere. We shall find that, when the distance exceeds a very moderate amount, we need to introduce a variable for the force of gravity which decreases with the square of the distance. Finally, at a height still greater, it is necessary to allow for the perturbing influence of the sun and of other gravitating bodies. In cases of this kind, the practical mathematician relinquishes the problem as insoluble and adopts the short and easy method of finding the theoretical velocity of a particle to infinity.

In this particular instance, it will be seen that the value of our useful empirical formula is, within a very limited range, very great; beyond that, for another limited range, it will answer fairly well, subject to a manageable correction; and finally it breaks down entirely, and the only way of obtaining the faintest clue to what really happens is by a method of approximation.

The errors are cumulative and after a certain limit entirely destroy the validity of the formula.[5]

This example shows us, lying on the surface of our subject, that one factor which limits the applicability of mathematical formulæ to actual entities is the magnitude of the terms and, particularly, of the simple fundamental variables space and time. Whenever a formula is applied beyond the ordinary limits of space and time, its value is doubtful, and if this extrapolation be carried out indefinitely, the applicability is entirely destroyed. While the results are still true as an ideal construction, considered as pure mathematics, they cease to have any bearing on actual perceptual existence.

This property can be illustrated by any physical formula applied to concrete problems. The simple relation between the power and weight in a lever and the distance of each from the fulcrum, will, as a rule, work without serious correction for a bar five feet long. For a bar 50 feet long corrections would be required for the weight of the lever and its lack of rigidity. For a bar 500 feet long, the formula would be useless.

The law of sines is applicable to simple cases of refraction in prisms and lenses, but, though still ideally true, it gives us little assistance in unraveling the interesting natural phenomena of mirage and fata morgana.

Joule's formula, expressing mathematically the first law of thermodynamics, is theoretically (except for the experimentally determined magnitude of J) an allotropic form of that most certain of all physical axioms, the conservation of energy; but it is a truism to say that the limits within which it will express the amount of heat that we can actually obtain in a measurable form, are almost vanishingly small.

From instances like these, which can be multiplied indefinitely, we can infer that, both by reason of the abstractions contained in mathematical formulæ as such and by reason of the complexity of the perceptual world to which they are applied, the applicability of each and every calculation is strictly limited, and, if its terms be multiplied indefinitely, it must necessarily break down and cease to provide any recognizable correspondence with objective fact.

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[5] I do not know whether the problem is insoluble, but any one who would calculate the velocity necessary to carry a particle into the sphere of gravitation of the sun, would introduce a large correction into speculations concerning the escape of planetary atmospheres which are now based on the theoretical "velocity to infinity." (Cf. papers by Dr. Johnson Stoney, *Astrophysical Journal*, 1898, and elsewhere.)

In the whole range of mathematical physics, there can not be found a single calculation which can be trusted to be accurate if carried too far from actual objective fact and from the possibility of verification. This indeed is the method of all science. Our abstractions, our calculations, must continually be brought into contact with the new particulars of fact which have been deduced from the abstractions. So far as this is done and so far as this is possible, theoretical questions concerning the necessary limits of mathematical reasoning do not arise in practical work. The limits are plain and obvious and inherent in the character of the work. The existence of facts of observation and experiment by which the theoretical conclusions of mathematical deduction can be tested and verified provides a practical check which effectually prevents undue and illegitimate extensions. It is only when these methods are applied to hypothetical happenings in the far distant realms of time and space, where there is no possibility of direct verification, that the examination of theoretical limits accomplishes, not only the intellectual purpose of aiding the comprehension of the intrinsic nature of scientific method, but the practical, every-day, useful work of providing a wholesome criticism and check on illegitimate speculation which takes upon itself the form and appearance of mathematical precision.

The danger of carrying deduction too far from the sphere of direct verification is recognized in discussions concerning the validity of extrapolation. All prediction is in one sense extrapolation, and all extrapolation possesses a certain theoretical uncertainty; but it is generally recognized that, within certain limits, this method is allowable.

One principle guiding the range of such deductions is found in the difference between rational and empirical formulæ. Extrapolation is much more valid with the former; but, as we have already seen, this distinction is merely one of degree. All but a very few have some elements of empirical or observational uncertainty. The well-known fact that extrapolations are regarded with considerable suspicion by practical scientists, and that such have so often proved mistaken, well illustrates the contention of this thesis.[6] Indeed, this well-known rule can be regarded as merely a particular case of the more general principle, that the applicability of all mathematico-physical formulæ is bound by ordinary limits to the magnitude of the terms, and particularly of those fundamental variables space and time.

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[6] Chemists will remember a striking instance of this in the determination of the temperature of liquid hydrogen. This was at first determined by Professor Dewar many degrees wrong by an extrapolation of an empirical conductivity formula.

So far we have been concerned only with simple mathematico-physical formulæ. It now remains to consider the bearing of these principles on more complicated forms of such deductive reasoning.

The complexity of a mathematical formula may be due to one or both of two possible causes. Its complexity may be occasioned by the process of successive approximation. A simple ideal formula, like that representing the fall of a body under gravity, may have added to it a number of corrections in order to enable it to calculate with greater accuracy the actual happenings in a world where such ideal conditions do not prevail. Such a formula bears on the surface evidence of its empirical character, and obviously will not admit of extensive extrapolation.

Another type of complex formula derives its complexity from deduction from more fundamental principles. Its theoretical certainty then depends on the foundations and on the process of deduction. For the range of this type no general rules can be assigned. This can only be determined empirically by a careful examination of its pedigree. To examine this for any particular formula is a problem which belongs to mathematics rather than to methodology.

Leaving this question of theoretical certainty, and confining our attention to practical applicability, it can readily be seen that a complex formula contains sources of error inherent in its own complexity. No detailed acquaintance with practical mathematics is requisite to perceive that every separate variable introduces an additional source of error, and that the error is increased as the grouping of the variables become more and more complex. A slight error in a factor of a complex differential produces a more than proportionate error in the result. It therefore follows that, assuming the same degree of theoretical certainty, the practical value and range of a mathematical formula decreases with each increase of complexity. A complex differential is not inherently superior to a simple formula, but is a useful instrument of investigation where no simple formula is available. Every separate variable introduces a new possibility of error, as does also the more complicated relations of the factors involved. For the purpose of obtaining an approximate solution to such a cosmic problem as the age of the earth, methods dependent on complex differentials are, other things being equal, inferior to those based on simple arithmetic.

At this point the present investigation ends. The fallacies inherent in the illicit mathematical treatment of cosmic problems have been traced to their theoretical source. In so doing, we have glanced at a number of theoretical questions and thrown out from time to time suggestions on which varying stress has been laid. It will now be convenient to sum up the main conclusions.

We have seen that, whatever may be its philosophical meaning, mathematical reasoning is an ideal construction, a series of abstractions, the validity and *a priori* truth of which is strictly limited to the conceptual sphere.

It is also clear that common knowledge and experimental science are a system of truths, largely based on observation and experiment, belonging to a perceptual series of an entirely different order.

All applications of mathematical reasoning to physical problems or to actual existence of any kind are, therefore, a more or less conscious fitting of the conceptual to the perceptual system. This process of fitting is subject to certain theoretical sources of error. These, which are commonly avoided by the continual contact of theoretical deductions with new detailed fact, are of special practical interest when such direct verification is only partially possible. Of these sources of error the following are of special interest: (1) the validity of mathematico-physical formulæ is dependent on the range and theoretical certainty of the physical or other principles involved in the premises; (2) it is also dependent on the completeness with which the conditions relevant to any particular problem have been abstracted from reality; (3) both these sources of error are increased with the complexity of the formulæ and with the magnitude of the substituted terms; (4) in cases of extensive extrapolation, such formulæ are liable to give not only minor errors, but results which bear no recognizable relation to concrete reality.

The essence of a methodological investigation lies in the possibility of concrete application. This little essay is an attempt to bring into closer relations methodology in the abstract and scientific method in the concrete. Though to the philosopher there may in the preceding pages appear to be but little of striking novelty, yet if these simple principles be fearlessly applied, it is clear that they will necessitate the reconsideration of much that has passed current as the outcome of scientific precision. Methods of finding the age of the earth by secular cooling, and much other cosmological speculation, appear in its true light as doubtful theory. Indeed, all acquainted with the multitudinous applications of mathematical reasoning to the complexity of the universe around us can, for themselves, find countless other examples.

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